10. I. V. Nemchinov, V. V. Svetsov, and V. V. Shuvalov, "Solution of the problem of the propagation of strong, intensely radiating shock waves in air by averaging the equation of radiation transfer." Low-Temperature Plasma in Space and Ion Earth, Collection of Articles [in Russian], VAGO, Moscow (1977), pp. 47-89.
11. V. I. Bergel'son, T. V. Loseva, and I. V. Nemchinov, "Numerical calculation of the problem of propagation of a plane subsonic radiation wave in a gas by averaging the equation of radiation transfer," in: Numerical Methods in the Physics of Plasma [in Russian], Nauka, Moscow (1977), pp. 234-236.
12. I. V. Nemchinov and V. V. Svetsov, "Calculation of the development of a laser explosion in air taking into account radiation," Zh. Prikl. Mekh. Tekh. Fiz., No. 4, 26-32 (1977).
13. S. Chandrasekar, Radiant Energy Transfer [Russian translation], IL, Moscow (1953).
14. P. N. Stevens, "Use of the discrete ordinates method in radiative shielding calculations," Nucl. Eng. Design, 13, 395-408 (1970).
15. B. C. Hankin, J. G. Hill, and A. G. P. Warham, "Numerical solution of the unsteady radiative transfer equation," J. Quant. Spectrosc. Radiat. Transfer, 11, 949-962 (1971).
16. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problems, Wiley (1967).
17. A. A. Samarskii and Yu. P. Popov, Difference Methods in Gas Dynamics [in Russian], Nauka, Moscow (1975).
18. G. S. Romanov, K. L. Stepanov, and S. I. Kas'kova, et al., "Spectral absorption coefficients of a high-temperature aluminum plasma," Otchet po NIR: Plasma [in Russian], NII PFP, Minsk, Gos. reg. No. B771439.

SOME EXACT SOLUTIONS TO EQUATIONS OF TRANSIENT FLOW WITH SUCTION FOR A VISCOUS FLUID

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Self-adjoint asymptotic solutions to the equations of flow are constructed for a viscous fluid near a permeable plane boundary.

We consider the problem of transient plane flow of an incompressible power-law non-Newtonian fluid near an infinitely large permeable wall in the plane of the x axis (Fig. 1). The fluid is uniformly sucked through the wall at a velocity $V_0(t)$. At the instant of time $t=0$ the wall is suddenly set in motion at a velocity $U_0(t)$ in the direction of the x axis [1].

We will consider only asymptotic solution, i.e., assume that all derivatives are $d/dx \equiv 0$. At infinity we let the velocity be not zero, as is usually done, but finite [2]. Under these assumptions, the equations of motion for a power-law fluid become

$$\frac{\partial v_1}{\partial t} + V_0(t)\frac{\partial v_1}{\partial y} = \frac{mn}{\rho}\left(\frac{\partial v_1}{\partial y}\right)^{n-1}\frac{\partial^2 v_1}{\partial y^2}, \tag{1}$$

$$\frac{dV_0}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

with the boundary conditions for the components of velocity and pressure

$$v_1 = v_2 = 0 \quad \text{at} \quad t=0,\ y>0, \tag{3}$$

$$v_1 = U_0(t),\ v_2 = V_0(t),\ p = p_0(t) \quad \text{at} \quad y=0,\ t>0. \tag{4}$$

We will henceforth deal only with the case $|n| < 1$. From Eq. (2) and the boundary condition (4) we determine the pressure

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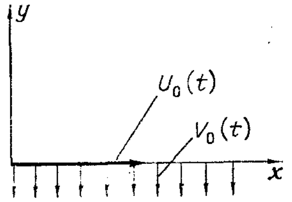

Fig. 1. System of coordinates and model of the wall.

$$p = p_0(t) - \rho \frac{dV_0}{dt} y.$$

The self-adjoint solution to Eq. (1) is found with the aid of the new variable

$$\eta = \left[\frac{\rho}{2mn(n+1)}\right]^{1/(n+1)} \frac{y - \int V_0(t)\,dt}{t^{1/(n+1)}}. \tag{5}$$

After a few transformations, we arrive at the ordinary differential equation

$$\frac{d^2v_1}{d\eta^2} + 2\eta\left(\frac{dv_1}{d\eta}\right)^{2-n} = 0 \tag{6}$$

and the boundary conditions

$$v_1 = 0 \quad \text{at} \quad \eta \to \infty, \tag{7}$$

$$v_1 = U_0(t) \quad \text{at} \quad \eta = \Phi(t,\ n),\ t > 0, \tag{8}$$

where

$$\Phi(t,\ n) = -\left[\frac{\rho}{2mn(n+1)}\right]^{1/(n+1)} \frac{\int V_0(t)dt}{t^{1/(n+1)}}.$$

The general solution to Eq. (6) is

$$v_1 = (1-n)^{1/(n-1)} \int (\eta^2 - C_1)^{1/(n-1)} d\eta + C_2. \tag{9}$$

A similar self-adjoint solution in form (9) has been obtained in another study [3] but only for specific values of $U_0(t)$ and $V_0(t)$, viz., $U_0 = \text{const}$ and $V_0 \sim t^{-\frac{n}{1+n}}$. The final form of the solution depends on the value of n and the sign of constant C_1.

We will take a specific value of n and will define the sign of C_1 as

$$C_1 = \pm a^2.$$

After the constant C_2 has been determined, let the solution be sought in the form

$$v_1 = \varphi(\eta,\ a).$$

Determination of the constant a reduces to resolution of a certain relation

$$U_0(t) = \varphi^*[t,\ V_0(t),\ a]. \tag{10}$$

In order to ensure self-adjointness of the solution, it is necessary to obtain the value of a from relation (10) as a constant. Obviously, the condition of constancy of a will be satisfied not for arbitrary functions $U_0(t)$ and $V_0(t)$ but only for these functions given in the form (10). This imposes a constraint on the form of function $U_0(t)$ with an arbitrary function $V_0(t)$ or vice versa. This then narrows down appreciably the class of possible solutions to the problem (6)-(8). The specific solution will depend on $U_0(t)$ and $V_0(t)$, viz.,

$$v_1 = v_1(t,\ y,\ U_0,\ V_0).$$

Let us consider a few special cases.

1. $n = 1/3$. After integrating the expression (9) and satisfying the boundary conditions (7)-(8), we write the solution in the form of a system where one of the parameters $U_0(t)$ or $V_0(t)$ is arbitrary and the other is related to it according to expression (10), and thus determine the constant C_1:

$$\begin{aligned} v_1 &= 1.5^{1.5} C_1^{-1}\left[1 - \frac{\eta}{\sqrt{\eta^2 - C_1}}\right], \\ U_0(t) &= 1.5^{1.5} C_1^{-1}\left[1 - \frac{\Phi(t,\ 1/3)}{\sqrt{\Phi^2(t,\ 1/3) - C_1}}\right]. \end{aligned} \tag{11}$$

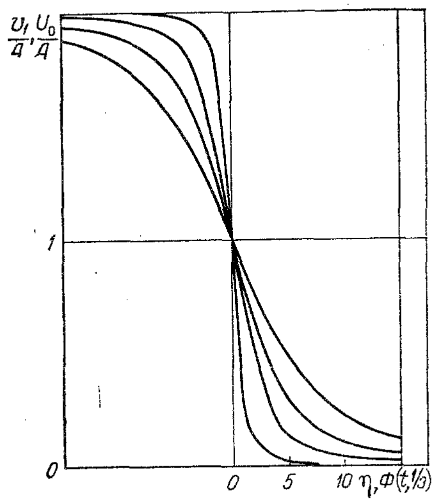

Fig. 2

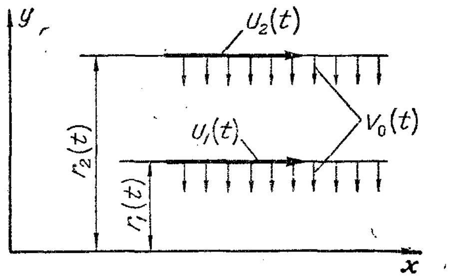

Fig. 3

Fig. 2. Longitudinal velocity of a power-law fluid ($n = 1/3$) and the wall: v_1/A and U_0/A in m/sec.

Fig. 3. Model of a channel with moving permeable walls.

When $C_1 = a^2$, then solution (11) is valid for $|\Phi(t, 1/3)| > a$ and $|\eta| > a$. When $C_1 = -a^2$, then solution (11) is valid for any η. The quantities v_1/A and $U_0(t)/A$ with $A = -1.5^{1.5}a^{-2}$ have been plotted in Fig. 2 for $a = 1$, 3, 5, and 8.

2. $n = 2/3$. When $C_1 = -a^2$, then integration of expression (9) and the boundary conditions (7)-(8) yield [4]

$$v_1 = 27\left[\frac{\eta}{4a^2(\eta^2+a^2)^2} + \frac{3\eta}{8a^4(\eta^2+a^2)} + \frac{3}{8a^5}\operatorname{arctg}\frac{\eta}{a} - \frac{3\pi}{16a^5}\right],$$

$$U_0(t) = 27\left\{\frac{\Phi(t,\ 2/3)}{4a^2[\Phi(t,\ 2/3)+a^2]^2} + \frac{3\Phi(t,\ 2/3)}{8a^4[\Phi^2(t,\ 2/3)+a^2]} + \frac{3}{8a^5}\operatorname{arctg}\frac{\Phi(t,\ 2/3)}{a} - \frac{3\pi}{16a^5}\right\}.$$

When $C_1 = a^2$, then the solution is obtained in the form

$$v_1 = -27\left[\frac{\eta}{4a^2(a^2-\eta^2)^2} + \frac{3\eta}{8a^4(a^2-\eta^2)} + \frac{3}{16a^5}\ln\left|\frac{a+\eta}{a-\eta}\right|\right],$$

$$U_0(t) = -27\left\{\frac{\Phi(t,\ 2/3)}{4a^2[a^2-\Phi^2(t,\ 2/3)]^2} + \frac{3\Phi(t,\ 2/3)}{8a^4[a^2-\Phi^2(t,\ 2/3)]} + \frac{3}{16a^5}\ln\left|\frac{a+\Phi(t,\ 2/3)}{a-\Phi(t,\ 2/3)}\right|\right\}. \tag{12}$$

Solution (12) has singularities at the points $\eta = \pm a$. Since $|n| < 1$, these singularities will, according to expression (9), appear at positive values of C_1.

In an analogous manner we will treat the transient flow of an incompressible viscous fluid through an infinitely long flat channel with moving permeable walls (Fig. 3). The walls can move in two mutually perpendicular directions with $U_1(t)$, $U_2(t)$ along the x axis and $r_1(t)$, $r_2(t)$ along the y axis as functions of time.

On the basis of the same original simplifying assumptions, the equations of motion for a Newtonian fluid become

$$\frac{\partial v_1}{\partial t} + V_0(t)\frac{\partial v_1}{\partial y} = \nu\frac{\partial^2 v_1}{\partial y^2},$$
$$\frac{dV_0}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{13}$$

with the boundary conditions

$$v_1 = v_2 = 0 \text{ when } t = 0, \tag{14}$$

$$v_1 = U_1(t),\ v_2 = V_0(t),\ p = p_1(t) \text{ at } y = r_1(t), \tag{15}$$

$$v_1 = U_2(t),\ v_2 = V_0(t),\ p = p_2(t) \text{ at } y = r_2(t). \tag{16}$$

From Eqs. (13) we determine the pressure

$$p = -\rho \frac{dV_0}{dt} y + C(t).$$

Function C(t) will be determined from the conditions (15) and (16):

$$p_1(t) = -\rho \frac{dV_0}{dt} r_1(t) + C(t),$$
$$p_2(t) = -\rho \frac{dV_0}{dt} r_2(t) + C(t). \tag{17}$$

Having two conditions (17) for one unknown function C(t) makes the problem an indeterminate one, which is logical in the given formulation. Since the fluid is incompressible and the channel walls are infinitely large, it is necessary to satisfy conditions of coupling between the change in pressure and the vibration mode of the channel walls

$$p_2(t) - p_1(t) = -\rho \frac{dV_0}{dt} [r_2(t) - r_1(t)]. \tag{18}$$

When relation (18) is satisfied, then function C(t) will be determined uniquely and the pressure can be expressed as

$$p = p_1 - \rho \frac{dV_0}{dt} [y - r_1(t)] = p_2 + \rho \frac{dV_0}{dt} [r_2(t) - y].$$

Introduction of the new variable

$$\eta = \frac{y - \int V_0(t)\,dt}{\sqrt{4\nu t}}$$

reduces the problem of determining the asymptotic profile of the longitudinal velocity to a solution of the ordinary differential equation

$$\frac{d^2 v_1}{d\eta^2} + 2\eta \frac{dv_1}{d\eta} = 0, \tag{19}$$

for the conditions

$$v_1 = 0 \quad \text{as} \quad \eta \to \infty,$$
$$v_1 = U_1(t) \quad \text{at} \quad \eta = \frac{r_1(t) - \int V_0(t)\,dt}{\sqrt{4\nu t}},$$
$$v_1 = U_2(t) \quad \text{at} \quad \eta = \frac{r_2(t) - \int V_0(t)\,dt}{\sqrt{4\nu t}}.$$

The solution to Eq. (19) is

$$v_1(\eta) = A(1 - \operatorname{erf}\eta),$$

with the integration constant A. Self-adjointness of this solution is ensured by the relations

$$U_1(t) = A\left[1 - \operatorname{erf}\left(\frac{r_1 - \int V_0 dt}{\sqrt{4\nu t}}\right)\right],$$
$$U_2(t) = A\left[1 - \operatorname{erf}\left(\frac{r_2 - \int V_0 dt}{\sqrt{4\nu t}}\right)\right]$$

obtained upon satisfying the boundary conditions.

Therefore, the system

$$p = p_1 - \rho \frac{dV_0}{dt} (y - r_1) = p_2 + \rho \frac{dV_0}{dt} (r_2 - y),$$
$$\frac{v_1}{A} = 1 - \operatorname{erf}\left(\frac{y - \int V_0 dt}{\sqrt{4\nu t}}\right),$$

$$v_2 = V_0(t), \quad \frac{U_1}{A} = 1 - \operatorname{erf}\left(\frac{r_1 - \int V_0 dt}{\sqrt{4\nu t}}\right),$$

$$\frac{U_2}{A} = 1 - \operatorname{erf}\left(\frac{r_2 - \int V_0 dt}{\sqrt{4\nu t}}\right)$$

constitutes the solution to the given problem for a Newtonian fluid.

In the case of a non-Newtonian power-law fluid, too, the distribution of velocity and pressure in the channel will be determined by Eqs. (1) and (2) with the boundary conditions (14)-(16). With the new variable (5) we obtain the ordinary differential equation (6), which must be solved for the boundary conditions

$$v_1 = 0 \text{ as } \eta \to \infty, \quad v_1 = U_1(t) \text{ at } \eta = \Phi_1(t, n), \quad v_1 = U_2(t) \text{ at } \eta = \Phi_2(t, n),$$

where

$$\Phi_i(t, n) = \left[\frac{\rho}{2mn(n+1)}\right]^{1/(n+1)} \frac{r_i(t) - \int V_0(t)\, dt}{t^{1/(n+1)}}, \quad i = 1, 2.$$

The general solution to Eq. (6) appears in the form (9). In analogy to the preceeding problem of a plate in an unbounded fluid, the solution can be written as

$$p = p_1 - \rho \frac{dV_0}{dt}(y - r_1) = p_2 + \rho \frac{dV_0}{dt}(r_2 - y),$$

$$v_1 = \varphi(\eta, a), \quad v_2 = V_0(t),$$

$$U_1 = \varphi[\Phi_1(t, n), a], \quad U_2 = \varphi[\Phi_2(t, n), a].$$

The final form of the solution will depend on the value of exponent n and on the sign of constant $C_1 = \pm a^2$.

Analogous solutions can also be obtained for the problem where an incompressible viscous fluid moves between an arbitrary number of moving permeable boundaries.

NOTATION

x, y, rectangular coordinates; t, time coordinate; η, a dimensionless coordinate; m, n, ν, ρ, parameters characterizing the fluid; p, pressure in the fluid; v_1 and v_2, components of the fluid velocity along axes x and y, respectively; U_1 and U_2, velocities of the wall along the x axis; r_1 and r_2, displacements along the y axis; and V_0, suction velocity.

LITERATURE CITED

1. H. Schlichting, Boundary Layer Theory, McGraw-Hill (1968).
2. H. S. Carslaw and J. C. Jaegar, Conduction of Heat in Solids, Oxford Univ. Press (1959).
3. Z. P. Shul'man and B. M. Berkovskii, Boundary Layer of non-Newtonian Fluids [in Russian], Nauka i Tekhnika, Minsk (1966).
4. H. B. Dwight, Tables of Integrals and Other Mathematical Data, Macmillian (1961).